Novel Quantum States of the Rational Calogero Models Without the Confining Interaction

B. Basu-Mallick[1], Pijush K. Ghosh[2] and Kumar S. Gupta[3]

Theory Group
Saha Institute of Nuclear Physics
1/AF Bidhannagar, Calcutta - 700064, India

Abstract

We show that the N-particle A_{N-1} and B_N rational Calogero models without the harmonic interaction admit a new class of bound and scattering states. These states owe their existence to the self-adjoint extensions of the corresponding Hamiltonians, labelled by e^{iz} where $z \in R$ (mod 2π). It is shown that the new states appear for all values of N and for specific ranges of the coupling constants. Moreover, they are shown to exist even in the excited sectors of the Calogero models. The self-adjoint extension generically breaks the classical scaling symmetry, leading to quantum mechanical scaling anomaly. The scaling symmetry can however be restored for certain values of the parameter z. We also generalize these results for many particle systems with classically scale invariant long range interactions in arbitrary dimensions.

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[1]Email: biru@theory.saha.ernet.in
[2]Email: pijush@theory.saha.ernet.in
[3]Email: gupta@theory.saha.ernet.in

1. Introduction

The spectrum of the N-particle one dimensional A_{N-1} rational Calogero model without the confining term has traditionally been described in terms of scattering states alone [1]. It has however recently been shown [2] that the spectrum of such a system admits negative energy bound states for certain values of the system parameters. These bound states owe their existence to the self-adjoint extension [3, 4] of the "radial" part of the Hamiltonian. These self-adjoint extensions are labelled by e^{iz} where $z \in R$. The parameter z (mod 2π) classifies all possible boundary conditions for the system. Indeed, the corresponding result for the two-body case has been known for a long time [5], which has recently found applications in the context of black holes [6, 7, 8].

In the analysis presented in Ref. [2], the new bound states of the A_{N-1} rational Calogero model were found to exist only for $N = 3$ and 4 and for the zero "angular momentum" sector of the problem. This happened due to the fact that the wavefunction was required to vanish when any two particles coincide. Such a boundary condition allows to smoothly extend the wavefunction for a given particle ordering to the whole of the N-particle configuration space corresponding to all possible orderings. In this paper we observe that this boundary condition on the wavefunction can be relaxed while still keeping it square integrable for any given particle ordering. The wavefunction obtained in this way can then be extended to all other particle orderings by using the permutation symmetry, although not in a smooth fashion. As a consequence of the more general boundary condition, we are now able to find the new bound states for arbitrary values of particle number N. Even more remarkably, the self-adjoint extensions are now found to exist even in the higher "angular momentum" sectors of the problem.

The scattering state solutions of the A_{N-1} model in absence of the confining term was already found by Calogero in Ref. [1]. The range of the coupling there was such that the corresponding Hamiltonian $H_{A_{N-1}}$ was essentially self-adjoint. In this paper we have been able to extend that analysis for the parameter range where $H_{A_{N-1}}$ is not essentially self-adjoint but admits self-adjoint extensions. In particular, we have found a new class of scattering states for each choice of the self-adjoint parameter z. The phase shifts for these scattering states depend explicitly on choice of z. Moreover, for generic values of z, the phase shift is found to depend on the momentum as well.

The A_{N-1} Calogero model without the confining term belongs to a class of N-particle systems with long range interactions which are classically scale invariant [9-15]. The scale invariance manifests itself through the absence of dimensionfull coupling constants. It is thus natural to ask if an analysis similar to that in Ref. [2] applies to this wider class of systems, including the higher dimensional cases. In this paper we address this issue and show that these systems indeed admit a new class of bound and scattering states. These states appear due to the self-adjoint extensions of the corresponding Hamiltonians. As examples of this general formulation, we have analyzed the bound and scattering state sectors of the A_{N-1} and B_N Calogero models in one dimension. The D_N and C_N Calogero models in one dimension arise as special cases of the B_N model. In all these cases we have found that the new quantum states exist for arbitrary values of N and for suitable ranges of the coupling constants. Moreover, these states appear in the excited sectors as well. The Calogero-Marchioro model [12] in D dimensions has also been analyzed to illustrate our formalism in dimensions higher than one. It has been shown that the new states exist in this model as well with suitable restrictions on the system parameters.

The classically scale invariant systems considered here are such that $O(2,1)$ appears as their spectrum generating algebra [16, 17]. It is however known that the classical scaling symmetry may be broken in the presence of the self-adjoint extensions [18, 19]. We shall show that for the general class of systems considered here, the self-adjoint extensions typically break the scale invariance at the quantum level. This is evident by the appearance of the momentum dependence of phase shifts in the scattering sector and by the appearance of bound states in certain cases. The underlying reason behind this is that the domain of self-adjointness of the associated Hamiltonian is not kept invariant by the scaling operator [19, 20, 21, 22]. Scale invariance at the quantum level is recovered only for special choices of the parameter z. The systems considered here thus provide further examples where scaling anomaly appears in quantum mechanics.

The organization of the paper is as follows. In Section 2 we provide a general analysis of N-particle scale invariant systems in one and higher dimensions. Using a suitable separation of variables, the effective Hamiltonian in the "radial" coordinate is obtained. The quantization of the effective Hamiltonian together with its self-adjoint extension is discussed in Section 3. It is shown that new bound and scattering states emerge naturally in this analysis. In Section 4, we discuss the application of this quantization method to several examples. These include A_{N-1}, B_N and D_N Calogero models in one dimension and Calogero-Marchioro model in higher dimensions. The restriction imposed on the system parameters necessary for the existence of the new quantum states is discussed in each case. We conclude the paper in Section 5 with some discussions.

2. Effective Hamiltonian for Scale Invariant Potentials

In this Section we show that the Hamiltonian of a class of many-particle systems with long range and classically scale invariant interactions can be related to the Hamiltonian of a single particle with inverse-square interaction on the half line. Let us consider a system of N identical particles of mass m in one dimension whose Hamiltonian in units of $2m\hbar^{-2} = 1$ is given by

$$H = \sum_{i=1}^{N} \left[-\partial_i^2 + (\partial_i w)^2 - \partial_i^2 w \right], \tag{2.1}$$

where ∂_i denotes the derivative with respect to the position of the i^{th} particle. The function w appearing in Eqn. (2.1) depends on the particle coordinates and is assumed to satisfy the relation

$$w = -\ln G, \tag{2.2}$$

where G is a homogeneous function of degree d, i.e.

$$\sum_{i=1}^{N} x_i \partial_i G = dG. \tag{2.3}$$

Using Eqns. (2.1) - (2.3), we can show that the many-body interaction in H scales in the same way as the kinetic energy term. This implies that there are no dimensionfull coupling constants in the system, indicating that H is classically scale invariant. It may be noted that $O(2,1)$ appears as the spectrum generating algebra for H [16, 17, 15].

We would like to solve the Schrödinger's equation

$$H\psi = E\psi. \tag{2.4}$$

For this purpose, consider an ansatz of the form

$$\psi = e^{-w} P_k(x)\phi(r), \tag{2.5}$$

where $x \equiv (x_1, x_2, \ldots, x_N)$ and $\phi(r)$ is taken to be a function of the "radial" variable r defined by $r^2 = \sum_{i=1}^{N} x_i^2$. The function $P_k(x)$ is a homogeneous polynomial of the particle coordinates with degree $k \geq 0$, i.e.

$$\sum_{i=1}^{N} x_i \partial_i P_k = kP_k, \tag{2.6}$$

satisfying the equation

$$-\sum_{i=1}^{N} \partial_i^2 P_k + 2\sum_{i=1}^{N} \partial_i w \partial_i P_k = 0. \tag{2.7}$$

Note that for $k = 0$, P_0 as an arbitrary constant (which can be chosen to be unity without any loss of generality) is a valid solution of Eqn. (2.7). The solutions of Eqn. (2.7) for higher values of k are known only for a few one dimensional many-body systems. If in a given model the solutions of Eqn. (2.7) for nonzero values of k are not known, we shall restrict our analysis only to the $k = 0$ sector, with $P_0 = 1$. Substituting Eqn. (2.5) in (2.4) and using Eqns. (2.3), (2.6) and (2.7), we get

$$\left[-\frac{\partial^2}{\partial r^2} - (1+2\nu)\frac{1}{r}\frac{\partial}{\partial r} \right] \phi(r) = E\phi(r), \tag{2.8}$$

where

$$\nu = \frac{N}{2} - 1 + k + d. \tag{2.9}$$

Lets us now define the function $\chi(r)$ through the transformation

$$\phi(r) = r^{-(\frac{1}{2}+\nu)}\chi(r). \tag{2.10}$$

Substituting Eqn. (2.10) in Eqn. (2.8) we get

$$\tilde{H}\chi(r) \equiv \left[-\frac{\partial^2}{\partial r^2} + \frac{\nu^2 - \frac{1}{4}}{r^2} \right] \chi(r) = E\chi(r). \tag{2.11}$$

The operator $\tilde{H}$ in Eqn. (2.11) defines the effective Hamiltonian of the system in the "radial" variable.

In deriving Eqn. (2.11) we did not make any assumption about translation invariance of the system. If the system is translationally invariant, then both $P_k(x)$ and $w(x)$ obey the same symmetry. In that case, we can still derive Eqn. (2.11) after eliminating the center of mass coordinate with r and ν now being given by

$$r^2 = \frac{1}{2N} \sum_{i,j=1}^{N} (x_i - x_j)^2 , \tag{2.12}$$

$$\nu = \frac{N-1}{2} - 1 + k + d. \tag{2.13}$$

Comparing Eqns. (2.9) and (2.13), we find that these two expressions differ by a contribution coming from the center of mass of the system.

In this paper we shall discuss both the scattering and bound states sectors of the Hamiltonian H, or equivalently, of the effective Hamiltonian $\tilde{H}$. For the purpose of discussion of the bound states, we now find the measure for which the eigenfunctions $\chi(r)$ would be square integrable. Using Eqns. (2.2), (2.5) and (2.10), we get

$$\psi = G P_k(x) r^{-(\frac{1}{2}+\nu)} \chi(r). \tag{2.14}$$

Note that r takes values in the positive real axis. For translationally invariant systems, the Hamiltonian H can equivalently be described using $(N-1)$ dimensional hyperspherical coordinates with the radius r and the $(N-2)$ angular variables Ω_i. In this case, both G and $P_k(x)$ are translationally invariant homogeneous function of degree d and k respectively. The wavefunction $\psi(x)$ can thus be factorized as $\psi(x) = \xi(r)F(\Omega)$ where

$$\xi(r) = r^{d+k-\nu-\frac{1}{2}} \chi(r), \tag{2.15}$$

and $F(\Omega)$ is a function of the angular variables Ω_i alone. Using the Eqns. (2.13) and (2.15), we see that

$$\int_0^\infty \xi^*(r)\xi(r) r^{(N-2)} dr = \int_0^\infty \chi^*(r)\chi(r) dr, \tag{2.16}$$

which implies that $\chi(r) \in L^2[R^+, dr]$. For translationally non-invariant systems, the Hamiltonian H can be described using N dimensional hyperspherical coordinates. The same line of argument as above using Eqn. (2.9) instead of Eqn. (2.13) again gives $\chi(r) \in L^2[R^+, dr]$.

We have so far restricted our attention where the space dimension is one. It can easily be shown that the above derivation is also valid in D space dimensions. In this case the particle coordinates would be D dimensional. Thus substituting $x_i \to \mathbf{r}_i$ and $\partial_i \to \nabla_i$ in Eqn. (2.1) and following the subsequent analysis, we get the same effective Hamiltonian $\tilde{H}$ with ν being given by

$$\nu = \frac{(N-1)D}{2} - 1 + k + d \tag{2.17}$$

for translationally invariant systems and

$$\nu = \frac{ND}{2} - 1 + k + d \tag{2.18}$$

for translationally non-invariant systems. Note that Eqns. (2.17) and (2.18) differ by a contribution coming from the center of mass of the system.

3. Quantization of the Effective Hamiltonian

In this Section we shall discuss the quantization of the effective Hamiltonian in Eqn. (2.11) [5]. We shall present the analysis in the scattering sector directly using von Neumann's theory of deficiency indices [3] when $\nu^2 > 0$. This will lead to an alternate derivation of the bound state solutions for this system as well. The operator $\tilde{H}$ belongs to a general class of objects known as unbounded linear differential operators on a Hilbert space [3]. We shall first outline some properties of these operators which are needed for our purpose.

Let T be an unbounded differential operator acting on a Hilbert space $\mathcal{H}$ and let $D(T)$ be the domain of T. The inner product of two element $\alpha, \beta \in \mathcal{H}$ is denoted by (α, β). Let $D(T^*)$ be the set of $\phi \in \mathcal{H}$ for which there is a unique $\eta \in \mathcal{H}$ with $(T\xi, \phi) = (\xi, \eta)\ \forall\ \xi \in D(T)$. For each such $\phi \in D(T^*)$, we define $T^*\phi = \eta$. T^* then defines the adjoint of the operator T and $D(T^*)$ is the corresponding domain of the adjoint. The operator T is called symmetric or Hermitian iff $(T\phi, \eta) = (\phi, T\eta)\ \forall\ \phi, \eta \in D(T)$. The operator T is called self-adjoint iff $T = T^*$ *and* $D(T) = D(T^*)$.

We now state the criterion to determine if a symmetric operator T is self-adjoint. For this purpose let us define the deficiency subspaces $K_\pm \equiv \mathrm{Ker}(i \mp T^*)$ and the deficiency indices $n_\pm(T) \equiv \dim[K_\pm]$. Then T falls in one of the following categories:
1) T is (essentially) self-adjoint iff $(n_+, n_-) = (0, 0)$.
2) T has self-adjoint extensions iff $n_+ = n_-$. There is a one-to-one correspondence between self-adjoint extensions of T and unitary maps from K_+ into K_-.
3) If $n_+ \neq n_-$, then T has no self-adjoint extensions.

We now return to the discussion of the effective Hamiltonian $\tilde{H}$. This is an unbounded differential operator defined in R^+. $\tilde{H}$ is a symmetric operator on the domain $D(\tilde{H}) \equiv \{\phi(0) = \phi'(0) = 0,\ \phi,\ \phi'$ absolutely continuous, $\phi \in \mathrm{L}^2(dx)\}$. We would next like to determine if $\tilde{H}$ is self-adjoint, for which we treat the cases $\nu \neq 0$ and $\nu = 0$ separately.

3.1 $\nu \neq 0$

The deficiency indices $n_\pm$ are determined by the number of square-integrable solutions of the equations

$$\tilde{H}^*\phi_\pm = \pm i\phi_\pm, \tag{3.1}$$

respectively, where $\tilde{H}^*$ is the adjoint of $\tilde{H}$. Note that $\tilde{H}^*$ is given by the same differential operator as $\tilde{H}$. From dimensional considerations we see that the r.h.s. of Eqn. (3.1) should be multiplied with a constant with dimension of length^{-2}. We shall henceforth choose the magnitude of this constant to be unity by appropriate choice of units. The solutions of Eqn. (3.1) are given by

$$\phi_+(r) = r^{\frac{1}{2}} H_\nu^{(1)}(re^{i\frac{\pi}{4}}), \tag{3.2}$$

$$\phi_-(r) = r^{\frac{1}{2}} H_\nu^{(2)}(re^{-i\frac{\pi}{4}}), \tag{3.3}$$

where H_ν's are Hankel functions [23]. The functions $\phi_\pm$ are bounded as $r \to \infty$. When $r \to 0$, they behave as

$$\phi_+(r) \;\to\; \frac{i}{\sin\nu\pi}\left[\frac{r^{\nu+\frac{1}{2}}}{2^\nu}\frac{e^{-i\frac{3\nu\pi}{4}}}{\Gamma(1+\nu)} - \frac{r^{-\nu+\frac{1}{2}}}{2^{-\nu}}\frac{e^{-i\frac{\nu\pi}{4}}}{\Gamma(1-\nu)}\right], \tag{3.4}$$

$$\phi_-(r) \;\to\; \frac{i}{\sin\nu\pi}\left[-\frac{r^{\nu+\frac{1}{2}}}{2^\nu}\frac{e^{i\frac{3\nu\pi}{4}}}{\Gamma(1+\nu)} + \frac{r^{-\nu+\frac{1}{2}}}{2^{-\nu}}\frac{e^{i\frac{\nu\pi}{4}}}{\Gamma(1-\nu)}\right]. \tag{3.5}$$

We see that $\phi_\pm$ are not square integrable when $\nu^2 \geq 1$. In this case $\tilde{H}$ has deficiency indices (0,0) and is (essentially) self-adjoint on the domain $D(\tilde{H})$ [4]. On the other hand, both $\phi_\pm$ are square integrable when either $-1 < \nu < 0$ or $0 < \nu < 1$. We therefore see that for any value of ν in these ranges, $\tilde{H}$ has deficiency indices $(1,1)$. In this case, $\tilde{H}$ is not self-adjoint on the domain $D(\tilde{H})$ but admits self-adjoint extensions. The deficiency subspaces $K_\pm$ in this case are one dimensional and are spanned by the functions $\phi_\pm$. The unitary maps from K_+ into K_- are parameterized by e^{iz} where $z \in R$ (mod 2π). The operator $\tilde{H}$ is self-adjoint in the domain $D_z(\tilde{H}) = D(\tilde{H}) \oplus \{a(\phi_+(r) + e^{iz}\phi_-(r))\}$ where a is an arbitrary complex number [3, 4].

We would now like to obtain the spectrum of $\tilde{H}$ in the domain $D_z(\tilde{H})$. We start with the discussion of the scattering states given by the positive energy solutions of Hamiltonian $\tilde{H}$. For that we set $E = q^2$, where q is a real positive parameter. The general solution of Eqn. (2.11) can be written as

$$\chi(r) = r^{\frac{1}{2}}\left[a(q)J_\nu(qr) - b(q)J_{-\nu}(qr)\right] \tag{3.6}$$

where $a(q)$ and $b(q)$ are two as yet undetermined coefficients. J_ν in Eqn. (3.6) refers to the Bessel function of order ν [23]. Note that in the limit $r \to 0$,

$$\phi_+(r) + e^{iz}\phi_-(r) \to \frac{i}{\sin\nu\pi}\left[\frac{r^{\nu+\frac{1}{2}}}{2^\nu}\frac{(e^{-i\frac{3\nu\pi}{4}} - e^{i(z+\frac{3\nu\pi}{4})})}{\Gamma(1+\nu)} + \frac{r^{-\nu+\frac{1}{2}}}{2^{-\nu}}\frac{(e^{i(z+\frac{\nu\pi}{4})} - e^{-i\frac{\nu\pi}{4}})}{\Gamma(1-\nu)}\right] \tag{3.7}$$

and

$$\chi(r) \to \frac{r^{\nu+\frac{1}{2}}}{2^\nu}\frac{a(q)q^\nu}{\Gamma(1+\nu)} - \frac{r^{-\nu+\frac{1}{2}}}{2^{-\nu}}\frac{b(q)q^{-\nu}}{\Gamma(1-\nu)}. \tag{3.8}$$

If $\chi(r) \in D_z(\tilde{H})$, then the coefficients of of $r^{\nu+\frac{1}{2}}$ and $r^{-\nu+\frac{1}{2}}$ in Eqns. (3.7) and (3.8) must match. Comparing these coefficients we get

$$\frac{a(q)}{b(q)} = \frac{\sin(\frac{z}{2} + 3\pi\frac{\nu}{4})}{\sin(\frac{z}{2} + \pi\frac{\nu}{4})}q^{-2\nu}. \tag{3.9}$$

Next we calculate the S-matrix and the associated phase shift. In the limit $r \to \infty$, the leading term in the asymptotic expansion of $\chi(r)$ in Eqn. (3.6) is given by

$$\chi(r) \to \frac{1}{\sqrt{2\pi q}}e^{iqr}\left[a(q)e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}} - b(q)e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}\right] + \frac{1}{\sqrt{2\pi q}}e^{-iqr}\left[a(q)e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} - b(q)e^{-i(\nu-\frac{1}{2})\frac{\pi}{2}}\right]. \tag{3.10}$$

By dividing the coefficient of outgoing wave (e^{iqr}) by that of the incoming wave (e^{-iqr}), we obtain the the S-matrix and phase shift $\delta(q)$ as

$$S(q) \equiv e^{2i\delta(q)} = \frac{a(q)e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}} - b(q)e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}}{a(q)e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} - b(q)e^{-i(\nu-\frac{1}{2})\frac{\pi}{2}}}. \tag{3.11}$$

Finally, using Eqns. (3.9) and (3.11), we get

$$S(q) = e^{2i\delta(q)} = \frac{q^{-\nu}\sin(\frac{z}{2}+3\pi\frac{\nu}{4})e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}} - q^{\nu}\sin(\frac{z}{2}+\pi\frac{\nu}{4})e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}}{q^{-\nu}\sin(\frac{z}{2}+3\pi\frac{\nu}{4})e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} - q^{\nu}\sin(\frac{z}{2}+\pi\frac{\nu}{4})e^{-i(\nu-\frac{1}{2})\frac{\pi}{2}}}. \tag{3.12}$$

Let us now consider the bound state solutions of Eqn. (2.11). For any given value of ν in the allowed range the S-matrix in Eqn. (3.12) has a pole on the positive imaginary axis of the complex q-plane. Such a pole indicates the existence of a bound state for the effective Hamiltonian $\tilde{H}$. By taking $q = i\rho$ as the pole for the S-matrix in Eqn. (3.12), one can easily derive the corresponding bound state energy $E = -\rho^2$ as

$$E = -\left[\frac{\sin(\frac{z}{2}+3\pi\frac{\nu}{4})}{\sin(\frac{z}{2}+\pi\frac{\nu}{4})}\right]^{\frac{1}{\nu}}. \tag{3.13}$$

Thus we see that for a given value of ν within the allowed range, $\tilde{H}$ admits a single bound state with energy given by Eqn. (3.13). It may be noted that for a fixed ν, the bound state exists only for those values of z such that the quantity $\frac{\sin(\frac{z}{2}+3\pi\frac{\nu}{4})}{\sin(\frac{z}{2}+\pi\frac{\nu}{4})}$ in Eqn. (3.13) is positive [5]. Using Eqns. (3.6), (3.9) and (3.13), we see that the the corresponding bound state eigenfunction is given by

$$\chi(r) = Br^{\frac{1}{2}}H^{(1)}_{\nu}(i\sqrt{|E|}r), \tag{3.14}$$

where B is the normalization constant. It may be noted that the expression of the bound state wavefunction for $0 < \nu < 1$ has been discussed in the literature [5, 6, 2] and agrees with Eqn. (3.14), when restricted to the same range of ν.

We would like to remind the reader that when $\nu^2 \geq 1$, $\tilde{H}$ is essentially self-adjoint in the domain $D(\tilde{H})$. In this case, the scattering state solutions are given by

$$\chi(r) = a(q)r^{\frac{1}{2}}J_{\nu}(qr), \tag{3.15}$$

which is same as Eqn. (3.6) without the term proportional to $J_{-\nu}(qr)$. No bound states exist in this case.

3.2 $\nu = 0$

In this case, the solutions of the equations

$$\tilde{H}^*\psi_{\pm} = \pm i\psi_{\pm} \tag{3.16}$$

are given by

$$\psi_+(r) = r^{\frac{1}{2}}H^{(1)}_0(re^{i\frac{\pi}{4}}), \tag{3.17}$$
$$\psi_-(r) = r^{\frac{1}{2}}H^{(2)}_0(re^{-i\frac{\pi}{4}}) \tag{3.18}$$

respectively. $\psi_{\pm}$ are bounded functions as $r \to \infty$. In order to find their behaviour for small r, we first note that in the limit $r \to 0$,

$$J_0(r) = 1 + \mathcal{O}\left(r^2\right) \tag{3.19}$$
$$N_0(r) = \frac{2}{\pi}[\gamma - \ln 2 + \ln r] + \mathcal{O}\left(r^2 \ln r\right) \tag{3.20}$$

where γ is Euler's constant and N_0 is the Neumann function [23]. Using Eqns. (3.17), (3.19) and (3.20) we see that when $r \to 0$, $\psi_+(r)$ behaves as

$$\psi_+(r) \to \frac{2i}{\pi} r^{\frac{1}{2}} \ln r + r^{\frac{1}{2}} \left[\frac{1}{2} + \frac{2i}{\pi}(\gamma - \ln 2) \right]. \tag{3.21}$$

In the same limit, $\psi_-(r)$ behaves as

$$\psi_-(r) \to -\frac{2i}{\pi} r^{\frac{1}{2}} \ln r + r^{\frac{1}{2}} \left[\frac{1}{2} - \frac{2i}{\pi}(\gamma - \ln 2) \right]. \tag{3.22}$$

From Eqns. (3.21) and (3.22) we see that both $\psi_\pm(r)$ are square integrable functions [8]. $\tilde{H}$ therefore has deficiency indices (1,1) and the corresponding deficiency subspaces $K_\pm$ are again 1-dimensional, spanned by the functions $\psi_\pm(r)$. As before, the operator $\tilde{H}$ is not self-adjoint on $D(\tilde{H})$ but admits a one-parameter family of self-adjoint extensions labeled by e^{iz} where $z \in R$. It is self-adjoint in the domain $D_z(\tilde{H})$ which contains all the elements of $D(\tilde{H})$ together with elements of the form $a(\psi_+(r) + e^{iz}\psi_-(r))$ where a is an arbitrary complex number. [3, 4].

The scattering states associated with positive energy solutions of Eqn.(2.11) are given by

$$\chi(r) = r^{\frac{1}{2}} \left[a(q) J_0(qr) - b(q) N_0(qr) \right], \tag{3.23}$$

where $E = q^2$ and $a(q)$, $b(q)$ are two as yet undetermined coefficients. In order to find the ratio of these two coefficients, as before we use the fact that if $\tilde{H}$ has to be self-adjoint, the eigenfunction $\chi(r)$ must belong to the domain $D_z(\tilde{H})$. In the limit when $r \to 0$, we have

$$\chi(r) \to -\frac{2b(q)}{\pi} r^{\frac{1}{2}} \ln r + r^{\frac{1}{2}} \left[a(q) - \frac{2b(q)}{\pi} \ln q + \frac{2b(q)}{\pi}(\ln 2 - \gamma) \right]. \tag{3.24}$$

We now compare the coefficients of $r^{\frac{1}{2}}$ and $r^{\frac{1}{2}}\ln r$ in Eqn. (3.24) with those appearing in the expression of $e^{i\frac{z}{2}}\psi_-(r) + e^{-i\frac{z}{2}}\psi_+(r)$. Comparing the coefficients of $r^{\frac{1}{2}}\ln r$ we find

$$b(q) = -2\sin\frac{z}{2}. \tag{3.25}$$

Comparing the coefficients of $r^{\frac{1}{2}}$ and using Eqn. (3.25) we obtain

$$a(q) - \frac{2b(q)}{\pi} \ln q = \cos\frac{z}{2}. \tag{3.26}$$

Using Eqns. (3.25) and (3.26) and assuming that $z \neq 0$, we finally obtain

$$\frac{a(q)}{b(q)} = \frac{2}{\pi} \ln q - \frac{1}{2}\cot\frac{z}{2}. \tag{3.27}$$

Thus we find that for any generic value of the self-adjoint parameter $z \neq 0$, the ratio of $a(q)$ and $b(q)$ depends on the momentum q. We shall comment on the $z = 0$ case at the end of this Section.

For calculating the S-matrix and phase shift for the above mentioned scattering process, we consider the leading term in the asymptotic expansion of $\chi(r)$ of Eqn. (3.23) at $r \to \infty$ limit. This gives

$$\chi(r) \to \frac{1}{\sqrt{2\pi q}} e^{iqr} e^{-i\frac{\pi}{4}} \left[a(q) + ib(q)\right] + \frac{1}{\sqrt{2\pi q}} e^{-iqr} e^{i\frac{\pi}{4}} \left[a(q) - ib(q)\right]. \tag{3.28}$$

By dividing the coefficient of outgoing wave (e^{iqr}) by that of the incoming wave (e^{-iqr}) and using Eqn. (3.28), we obtain the the S-matrix and phase shift as

$$S(q) \equiv e^{2i\delta(q)} = e^{-i\frac{\pi}{2}} \frac{\frac{2}{\pi}\ln q - \frac{1}{2}\cot\frac{z}{2} + i}{\frac{2}{\pi}\ln q - \frac{1}{2}\cot\frac{z}{2} - i}. \tag{3.29}$$

When $z \neq 0$, the S-matrix in Eqn. (3.29) again has a single pole on the positive imaginary axis of the complex q-plane. Such a pole naturally indicates the existence of a bound states for the effective Hamiltonian $\tilde{H}$. By taking $q = i\mu$ as the pole for the S-matrix in Eqn. (3.29), we obtain the corresponding bound state energy $E = -\mu^2$ as

$$E = -\exp\left[\frac{\pi}{2}\cot\frac{z}{2}\right]. \tag{3.30}$$

The corresponding eigenfunction is given by

$$\chi(r) = Cr^{\frac{1}{2}} K_0\left(\sqrt{|E|}r\right) = C\frac{i\pi}{2} r^{\frac{1}{2}} H_0^{(1)}\left(i\sqrt{|E|}r\right), \tag{3.31}$$

where C is a constant and K_0 is the modified Bessel function [23].

This concludes our discussion regarding the quantization of the effective Hamiltonian $\tilde{H}$. We end this Section with the following general observations.

1) We have seen that for a given value of ν, each value of the parameter z (mod 2π) produces a different spectrum of $\tilde{H}$. The spectrum of $\tilde{H}$, and correspondingly the parameter space of the quantum theory is thus characterized by the pair (ν, z). Let us assume for the moment that $\nu \neq 0$. Consider now two pairs of parameters (ν, z) and (ν', z') corresponding to two different quantum theories. It is easily seen that the spectrum of these two theories are identical if $\nu' = -\nu$ and $z' = z + 2\pi\nu$. We therefore have an equivalence relation $(\nu, z) \sim (-\nu, z + 2\pi\nu)$ on the parameter space of $\tilde{H}$. The transformation $(\nu, z) \to (\nu', z')$ relating two different quantum theories is analogous to the duality symmetry in this system. In the case when $\nu = 0$, we automatically have $(\nu, z) = (\nu', z')$ for all values of z. We can thus say that the pair $(\nu = 0, z)$ defines the self-dual points in the parameter space.

2) As mentioned before, the classical system that we started with possesses scaling symmetry. However, in the presence of the self-adjoint extensions, the system admits bound state(s) and the phase shifts in the scattering sector depend explicitly on the momentum. These are indicative of the breakdown of the scaling symmetry at the quantum level. We first analyze this issue when $-1 < \nu \neq 0 < 1$. Let us consider the action of the scaling operator $\Lambda = \frac{-i}{2}(r\frac{d}{dr} + \frac{d}{dr}r)$ on an element $\phi(r) = \phi_+(r) + e^{iz}\phi_-(r) \in D_z(\tilde{H})$. In the limit $r \to 0$, we have

$$\Lambda\phi(r) \to \frac{1}{\sin\nu\pi}\left[(1+\nu)\frac{r^{\nu+\frac{1}{2}}}{2^{\nu}}\frac{\left(e^{-i\frac{3\nu\pi}{4}} - e^{i(z+\frac{3\nu\pi}{4})}\right)}{\Gamma(1+\nu)} + (1-\nu)\frac{r^{-\nu+\frac{1}{2}}}{2^{-\nu}}\frac{\left(e^{i(z+\frac{\nu\pi}{4})} - e^{-i\frac{\nu\pi}{4}}\right)}{\Gamma(1-\nu)}\right]. \tag{3.32}$$

In order for $\Lambda\phi(r) \in D_z(\tilde{H})$, we must have $\Lambda\phi(r) \sim C\phi(r)$ where C is a constant. However, the two terms on the r.h.s. of Eqn. (3.32) are multiplied by two different factors , i.e. $(1+\nu)$ and $(1-\nu)$. Due to the presence of these different multiplying factors, we see that $\Lambda\phi(r)$ in general does not belong to $D_z(\tilde{H})$. Scale invariance is thus broken at the quantum level for generic values of z. However, from Eqn. (3.32) it is clear that for special choice of $z = -\frac{\nu\pi}{2}$, $\Lambda\phi(r) \in D_z(\tilde{H})$ and the scaling symmetry is recovered [6]. In addition, we find that the scaling symmetry is also preserved at the quantum level when $z = -\frac{3\nu\pi}{2}$. For these choices of z, the bound states do not exist and the S matrix becomes independent of the momentum. The scaling symmetry thus is anomalously broken due to the quantization for generic values of z. In the case when $\nu = 0$, a similar analysis as above again shows that the self-adjoint extension generically breaks the scaling symmetry. For this case, the scale invariance can be recovered at the quantum level only for $z = 0$.

4. Examples

We have shown in the previous Section that the effective Hamiltonian $\tilde{H}$, and thus the Hamiltonian H, admits self-adjoint extensions when $-1 < \nu < 1$. As discussed before, ν is a function of several system parameters. The condition on ν implies that $\tilde{H}$ admits self-adjoint extensions only if the system parameters are suitably constrained. In this Section we shall consider several models with suitable ranges of parameters, such that the corresponding Hamiltonians admit self-adjoint extensions. The new bound and scattering states will be obtained by substituting the solutions of eigenvalue Eqn. (2.11) into Eqn. (2.14). As mentioned before, the effective Hamiltonian is essentially self-adjoint when $\nu^2 \geq 1$. This is the case considered by Calogero in Ref. [1] for the A_{N-1} model. In the examples discussed below we shall not consider the range $\nu^2 \geq 1$.

4.1 A_{N-1} Calogero Model

The Hamiltonian of the A_{N-1} Calogero Model without the harmonic term is given by

$$H_{A_{N-1}} = -\sum_i \frac{d^2}{dx_i^2} + (a^2 - \frac{1}{4}) \sum_{i \neq j} (x_i - x_j)^{-2}, \tag{4.1}$$

where $i, j = 1, 2, \cdots, N$. This Hamiltonian can be obtained from Eqns. (2.1) and (2.2) by choosing G as

$$G = \prod_{i<j} (x_i - x_j)^{a+\frac{1}{2}}, \tag{4.2}$$

which is a homogeneous function of degree $d = (a + \frac{1}{2})N(N-1)/2$. The Hamiltonian $H_{A_{N-1}}$ is translationally invariant and the parameter ν is determined from Eqn. (2.13) as

$$\nu = k + \left(a + \frac{1}{2}\right) \frac{N(N-1)}{2} + \frac{N-3}{2}. \tag{4.3}$$

For this model, it is known that Eq. (2.7) determining $P_k(x)$ is exactly solvable for any $k \geq 0$ [1, 9]. Below we shall discuss both bound and scattering state sectors of the Hamiltonian $H_{A_{N-1}}$ for arbitrary $k \geq 0$.

The bound state sector of this system was analyzed in Ref. [2] under the assumption that $a + \frac{1}{2} \geq 0$. In this case, the wavefunction vanishes in the limit when any two particle coordinates coincide, though the corresponding current might show a divergent behaviour. Such a boundary condition on the wavefunction is quite similar to what one encounters in the case of δ-function Bose gas with infinitely large value of two-body coupling constant [10, 24]. This type of boundary condition allows one to construct continuous eigenfunctions on the whole of configuration space by first solving the eigenvalue problem for a definite ordering of particles e.g. $x_1 \geq x_2 \geq \cdots \geq x_N$ and then smoothly extending it to the rest of the configuration space using permutation symmetry associated with identical particles. It is obvious from Eqn. (4.3) that for $a + \frac{1}{2} \geq 0$ and $N \geq 3$, ν is a positive definite quantity. Thus, the allowed ranges of ν for $H_{A_{N-1}}$ to have bound states is further restricted to $0 < \nu < 1$. Analyzing Eqn. (4.3) with the constraints on $a + \frac{1}{2}$ and ν discussed above, it was shown in Ref. [2] that the A_{N-1} Calogero model admits negative energy bound states only for $N = 3, 4$ and $k = 0$.

The bound state wavefunctions of $H_{A_{N-1}}$ obtained in Ref. [2] are normalizable, although as $r \to 0$, they have a singularity of the form $r^{-\nu - \frac{N-3}{2}}$. Note that the singularity at $r = 0$ corresponds to the case where all the particles coincide at the same point. This is different from the singularity arising from the coincidence of any two particles, which is avoided by imposing the constraint $a + \frac{1}{2} \geq 0$. However, from the viewpoint of self-adjoint extensions alone, there are no reasons a priori to distinguish between the singularity at $r \to 0$ limit and the singularity arising from the coincidence of any two particle coordinates. In view of this, we now let both these types of singularities to appear in the wavefunction. It may be noted that for both bound and scattering state solutions, the angular part of the total wavefunction must be square integrable. The angular part of the wavefunction receives a contribution from the factor G in Eqn. (4.2), which is singular when $a+\frac{1}{2} < 0$. Requiring square integrability of the angular part of the wavefunction puts the restriction that $a+\frac{1}{2} > -\frac{1}{2}$. A negative value for the parameter $a+\frac{1}{2}$ in this range leads to a singularity in the wavefunction resulting from the coincidence of any two particle coordinates. This restricts the range of a continuous eigenfunction within a region of configuration space corresponding to definite ordering of particles like $x_1 \geq x_2 \geq \cdots \geq x_N$. Using permutation symmetry, such an eigenfunction can be extended to the whole of phase space, although not in a smooth fashion. The bound state wavefunctions of $H_{A_{N-1}}$ thus obtained are completely normalizable for $a + \frac{1}{2} > -\frac{1}{2}$.

We now analyze the case where N is fixed while $a + \frac{1}{2}$ is allowed to vary subject to the restriction that $a + \frac{1}{2} > -\frac{1}{2}$. We note that since $-1 < \nu < 1$, we must have,

$$-\frac{N-1+2k}{N(N-1)} < a + \frac{1}{2} < -\frac{N-1+2k}{N(N-1)} + \frac{4}{N(N-1)}. \tag{4.4}$$

Imposing the condition that the upper bound on $a + \frac{1}{2}$ in Eqn. (4.4) should be greater than $-\frac{1}{2}$, we find that k is restricted as

$$k < \frac{1}{4}\left(N^2 - 3N + 10\right). \tag{4.5}$$

Thus, the allowed values of k are $0, 1, 2, \ldots, K \equiv \{(N^2 - 3N + 10)/4\}$. The symbol $\{x\}$ denotes the integral part of x if x is non-integer and is equal to $x - 1$ for integer x. For a fixed value of N, the self-adjoint extension is admissible when the coupling constant lies within the range

$$-\frac{1}{2} < a + \frac{1}{2} < \frac{5-N}{N(N-1)}. \tag{4.6}$$

Note that the upper bound in Eqn. (4.6) is negative for $N > 5$. This implies that the interaction in Eqn. (4.1) is repulsive for $N > 5$.

We have seen in the previous paragraph that for fixed value of N and variable coupling, the self-adjoint extension and, hence, new quantum states are admissible for several values of k. We now ask as to what are the allowed values of k when both N and $a + \frac{1}{2}$ are kept fixed. To this end, we introduce a quantity $\beta(a, N) = (a+\frac{1}{2})N(N-1)/2 + (N-3)/2$ so that $\nu = k + \beta(a, N)$. Note that for fixed values of a and N, β is fixed. Moreover, since $a + \frac{1}{2} > -\frac{1}{2}$, β is bounded from below, $\beta > -\beta_0 \equiv -\frac{1}{4}(N^2 - 3N + 6)$. It is now easy to see that with $-1 < \nu < 1$ and $k \geq 0$, number of allowed values of k for fixed β is at most 2. In order to find the specific values of k, we consider the cases $\nu \neq 0$ and $\nu = 0$ separately.

4.1.1 $\nu \neq 0$

We now use the constraint, $-1 < \nu \neq 0 < 1$, and present our results in the table below.

Bounds on β	Ranges of $a + \frac{1}{2}$	Allowed values of k
$\beta \geq 1$	$a + \frac{1}{2} \geq \frac{5-N}{N(N-1)}$	$\times$
$0 < \beta < 1$	$\frac{3-N}{N(N-1)} < a + \frac{1}{2} < \frac{5-N}{N(N-1)}$	0
$-1 < \beta < 0$	$-\frac{1}{N} < a + \frac{1}{2} < \frac{3-N}{N(N-1)}$	0, 1
$-2 < \beta < -1$	$-\frac{N+1}{N(N-1)} < a + \frac{1}{2} < -\frac{1}{N}$	1, 2
.	.	.
$-(l+1) < \beta < -l$	$-\frac{N+2l-1}{N(N-1)} < a + \frac{1}{2} < -\frac{N+2l-3}{N(N-1)}$	$l, l+1$
.	.	.
$-(K-1) < \beta < -(K-2)$	$-\frac{N+2K-5}{N(N-1)} < a + \frac{1}{2} < -\frac{N+2K-7}{N(N-1)}$	$K-2, K-1$
$-\beta_0 < \beta < -(K-1)$	$-\frac{1}{2} < a + \frac{1}{2} < -\frac{N+2K-5}{N(N-1)}$	$K-1, K$

It is curious to observe how only two successive values of k are allowed within a specific range of $a + \frac{1}{2}$, although any lower value than this k within the same range is not allowed.

When $\nu \neq 0$, using Eqns. (2.14), (3.14) and (4.2), we see that the bound state wavefunction for $H_{A_{N-1}}$ is obtained as

$$\psi = B\prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_k(x) r^{-\nu} H_\nu^{(1)}(i\sqrt{|E|}r). \tag{4.7}$$

The corresponding bound state energy E is given in Eqn. (3.13). Note that for fixed values of N, a and z, the system may admit two bound states corresponding to two possible values of k. Similarly, the scattering state solutions of $H_{A_{N-1}}$ obtained by using Eqns. (2.14), (3.6) and (4.2) are given by

$$\psi = \prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_k(x) r^{-\nu} \left[a(q)J_\nu(qr) - b(q)J_{-\nu}(qr)\right], \tag{4.8}$$

where $\frac{a(q)}{b(q)}$ is given in Eqn. (3.9).

4.1.2 $\nu = 0$

We now discuss the case for which $\nu = 0$, or equivalently $\beta = -k$. It follows from Eqn.(4.3) that $a + \frac{1}{2} = \frac{3-N-2k}{N(N-1)}$. Since $a + \frac{1}{2} > -\frac{1}{2}$, the allowed values of k are $0, 1, \ldots, \{(N^2 - 3N + 6)/4\}$. In this case, using Eqns. (2.14), (3.31) and (4.2) we see that the bound state wavefunction for the Hamiltonian $H_{A_{N-1}}$ is given by

$$\psi = C\frac{i\pi}{2}\prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_k(x) r^{-\nu} H_0^{(1)}\left(i\sqrt{|E|}r\right), \tag{4.9}$$

where the bound state energy E are given in Eqn. (3.30). Similarly, the scattering state solutions in this case are obtained using Eqns. (2.14, (3.23) and (4.2) as

$$\psi = \prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_k(x) r^{-\nu} \left[a(q)J_0(qr) - b(q)N_0(qr)\right], \tag{4.10}$$

where $\frac{a(q)}{b(q)}$ is given in Eqn. (3.27).

We end this Section with the following observations:
(i) Self-adjoint extension and, hence, the existence of the new quantum states is admissible in the A_{N-1} Calogero model for arbitrary N.
(ii) The self-adjoint extension for $k > 0$ is remarkable in the following sense. If a Hamiltonian admits self-adjoint extension, it usually occurs in the zero angular momentum sector of the Hamiltonian. We are not aware of any counter-example to this fact. For the $H_{A_{N-1}}$, k can be related to the angular-momentum eigenvalue of the Hamiltonian[11]. Considering the case $k \neq 0$ amounts to studying the model in the non-zero angular momentum sector of the model. Thus, we provide the first example in the literature where self-adjoint extension is admissible in non-zero angular momentum sector of a Hamiltonian.

4.2 B_N Calogero model

The Hamiltonian of the B_N Calogero model without the harmonic term is given by

$$\begin{aligned} H_{B_N} &= -\sum_{i=1}^{N}\frac{d^2}{dx_i^2} + g_1(g_1 - 1)\sum_{i\neq j}\left[(x_i - x_j)^{-2} + (x_i + x_j)^{-2}\right] \\ &\quad + g_2(g_2 - 1)\sum_{i=1}^{N} x_i^{-2}. \end{aligned} \tag{4.11}$$

The Hamiltonian in Eqn. (4.11) can be obtained from Eqns. (2.1) and (2.2) by choosing G as

$$G = \prod_{i<j} (x_i^2 - x_j^2)^{g_1} \prod_s x_s^{g_2}, \tag{4.12}$$

where $i, j, s = 1, 2, \cdots, N$. G in Eqn. (4.12) is a homogeneous function of degree $d = g_1 N(N-1) + g_2 N$. It is known that for the B_N Calogero model, Eqn. (2.7) admits solutions only for even $k \geq 0$ [9]. Moreover, the Hamiltonian H_{B_N} is not translationally invariant. Consequently, ν is determined from Eqn. (2.9) as

$$\nu = \frac{N}{2} - 1 + g_1 N(N-1) + g_2 N + 2k, \tag{4.13}$$

where $k = 0, 1, 2, \ldots$. Using the constraint $-1 < \nu < 1$, it can be shown that for fixed values of N, g_1 and g_2, there is at most one allowed value of k.

We encounter three kinds of singularities in the Hamiltonian (4.11). They occur whenever,
(i) any two particles coincide, i.e. $x_i \to x_j$,
(ii) any particle is at the position of the image of any other particle, i.e. $x_i \to -x_j$,
(iii) any particle is at the origin, i.e. $x_i \to 0$.
If we do not want the wavefunction to reflect singularities of these kinds, we have to put the restriction $g_1, g_2 \geq 0$. For such cases, the new states exists only in the $k = 0$ sector and for $N = 2$ and 3. Compared to the similar situation in the case of A_{N-1} Calogero model, $N = 4$ is not allowed for the present case. This may be attributed to the fact that B_N Calogero model is not translationally invariant.

In the more general case, following the discussion for the A_{N-1} Calogero model, we allow the wavefunction to have singularities while maintaining the square-integrability of the angular part, which demands that $g_1, g_2 > -\frac{1}{2}$. We would now like to find the allowed values of k when N is kept fixed but the couplings are allowed to vary. To this end, we first write g_1 and g_2 as,

$$g_1 = -\frac{1}{2} + \epsilon_1^2, \quad g_2 = -\frac{1}{2} + \epsilon_2^2, \tag{4.14}$$

so that ϵ_1 and ϵ_2 can take any nonzero real values. Using the constraint $-1 < \nu < 1$, we find that

$$\frac{1}{2} - \frac{2k}{N(N-1)} < \epsilon_1^2 + \frac{\epsilon_2^2}{N-1} < \frac{1}{2} - \frac{2k}{N(N-1)} + \frac{2}{N(N-1)}. \tag{4.15}$$

The upper bound in Eq. (4.15) restricts k as

$$k < \frac{1}{4} N(N-1) + 1. \tag{4.16}$$

The allowed values of k are given by $k = 0, 1, 2, \ldots, K \equiv \{\frac{1}{4} N(N-1) + 1\}$. When $k = K$, the l.h.s. of Eqn. (4.15) becomes negative if $\frac{1}{4} N(N-1)$ is not an integer. In that case we replace the lower bound in Eqn. (4.15) by 0. It is evident from the above analysis that for a fixed value of N, new states exist for the range of the couplings given by

$$0 < \epsilon_1^2 + \frac{\epsilon_2^2}{N-1} < \frac{1}{2} + \frac{2}{N(N-1)}, \tag{4.17}$$

which defines an elliptical region in the $\epsilon_1 - \epsilon_2$ plane. Since $g_1, g_2 > -\frac{1}{2}$, the lines given by $\epsilon_1 = 0$ and $\epsilon_2 = 0$ are excluded from this elliptic region. For different allowed values of k, this region naturally separates into disjoint elliptical shells defined by Eqn. (4.15). The outermost shell corresponds to $k = 0$ while the innermost one corresponds to $k = K$. There are no new quantum states for the values of the couplings corresponding to the boundary separating two consecutive shells.

When $\nu = 0$, the couplings satisfy the relation

$$\epsilon_1^2 + \frac{\epsilon_2^2}{N-1} = \frac{1}{2} + \frac{1-2k}{N(N-1)}. \tag{4.18}$$

The allowed values of k in this case are found to be $k = 0, 1, \ldots, \{(N^2 - N + 2)/4\}$. In general, a shell defined by Eqn. (4.15) contains an ellipse of the form (4.18) with the same value of k. However, when $\frac{1}{4}N(N-1)$ is an integer, the value of $\nu = 0$ cannot be achieved within the innermost shell corresponding to $k = K$.

When $\nu \neq 0$, the bound state wavefunction of the B_N model is given by

$$\psi = B \prod_{i<j} (x_i^2 - x_j^2)^{g_1} \prod_s x_s^{g_2} \tilde{P}_{2k}(x) r^{-\nu} H_\nu^{(1)}(i\sqrt{|E|}r), \tag{4.19}$$

where the bound state energy E is given by Eqn. (3.13). $\tilde{P}$ appearing in Eqn. (4.19) is a solution of Eqn. (2.7) for the B_N Calogero model. Similarly, the scattering state solutions are given by

$$\psi = \prod_{i<j} (x_i^2 - x_j^2)^{g_1} \prod_s x_s^{g_2} \tilde{P}_{2k}(x) r^{-\nu} \left[a(q) J_\nu(qr) - b(q) J_{-\nu}(qr)\right], \tag{4.20}$$

where $\frac{a(q)}{b(q)}$ is given by Eqn. (3.9). When $\nu = 0$, the bound state wavefunction is given by

$$\psi = C \frac{i\pi}{2} \prod_{i<j} (x_i^2 - x_j^2)^{g_1} \prod_s x_s^{g_2} \tilde{P}_{2k}(x) r^{-\nu} H_0^{(1)} \left(i\sqrt{|E|}r\right), \tag{4.21}$$

where the bound state energy E are given in Eqn. (3.30). Similarly, the scattering state solutions are given by

$$\psi = \prod_{i<j} (x_i^2 - x_j^2)^{g_1} \prod_s x_s^{g_2} \tilde{P}_{2k}(x) r^{-\nu} \left[a(q) J_0(qr) - b(q) N_0(qr)\right], \tag{4.22}$$

where $\frac{a(q)}{b(q)}$ is given by Eqn. (3.27).

4.2.1 D_N Calogero Model

The D_N Calogero model can be obtained from the B_N model by putting $g_2 = 0$, i.e. $\epsilon_2^2 = \frac{1}{2}$. Using the expression for ν in (4.13) and the constraint $-1 < \nu < 1$, we find,

$$-\frac{N+4k}{2N(N-1)} < g_1 < -\frac{N+4k}{2N(N-1)} + \frac{2}{N(N-1)}. \tag{4.23}$$

The constraint $g_1 > -\frac{1}{2}$ restricts k as

$$k \leq \frac{1}{4}(N^2 - 2N + 4). \tag{4.24}$$

Thus, the quantum number k can take any integral values, $k = 0, 1, 2, \ldots, K \equiv \{\frac{1}{4}(N^2 - 2N + 4)\}$. It is clear that for a fixed value of N, new states exist for the range of g_1 given by

$$-\frac{1}{2} < g_1 < \frac{4 - N}{2N(N-1)}. \tag{4.25}$$

Note, however, that the points given by $g_1 = -\frac{N+4k}{2N(N-1)}$ belonging to the range in Eqn. (4.25) are excluded. For fixed values of g_1 and N, the number of allowed values of k is at most one. When $\nu \neq 0$, the bound and scattering state wavefunction in this case are obtained from Eqns. (4.19) and (4.20) respectively by putting $g_2 = 0$ in them. When $\nu = 0$, $g_1 = -\frac{N+4k-2}{2N(N-1)}$ with $k = 0, 1, \ldots, \{(N^2 - N + 2)/4\}$. The bound and scattering states wavefunctions in this case are obtained from Eqns. (4.21) and (4.22) respectively by putting $g_2 = 0$ in them.

Finally, as in the case of A_{N-1} Calogero model, we have the remarkable results that for suitable ranges of couplings, the B_N and D_N Calogero models admit self-adjoint extensions and consequently, new scattering and bound states for arbitrary N. These new states exist for certain non-zero angular momentum sector also. For fixed values of N and the relevant couplings, $H_{A_{N-1}}$ admits a maximum of two bound states corresponding to at most two possible values of k. On the other hand, under similar conditions, B_N and D_N Calogero models admit only one bound state. It may be noted that the Hamiltonian of the C_N Calogero model is identical to H_{B_N} except that the last term in Eqn. (4.11)is replaced by $g_3(g_3-1)\sum_{i=1}^{N}(2x_i)^{-2}$ where g_3 is a constant. The analysis presented above can be extended to the C_N Calogero model as well.

4.3 Calogero-Marchioro Model : A D Dimensional Example

All the models that have been discussed so far are one dimensional. We now give an example of a D dimensional many-particle system, known as the Calogero-Marchioro model in the literature [12, 13]. The Hamiltonian for this model is given by

$$H_{CM} = -\sum_{i=1}^{N} \boldsymbol{\nabla}_i^2 + g(g + D - 2)\sum_{i\neq j}\frac{1}{\mathbf{r}_{ij}^2} + g^2 \sum_{i\neq j\neq k}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{\mathbf{r}_{ij}^2\mathbf{r}_{ik}^2}, \tag{4.26}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The Hamiltonian in Eqn. (4.26) can be obtained from D-dimensional analogue of Eqn. (2.1) by choosing

$$G = \prod_{i\neq j}\mathbf{r}_{ij}^g. \tag{4.27}$$

G is a homogeneous function of degree $d = gN(N-1)/2$. H_{CM} contains both two-body and three-body interaction terms. For $D = 1$, the three-body interaction term vanishes identically and H_{CM} reduces to the the $H_{A_{N-1}}$. For $D \geq 2$, although infinitely many exact solutions corresponding to the radial excitations are known, the model is not exactly solvable. In particular, Eq. (2.7) determining solutions for P_k is not known for $k > 0$. We will henceforth restrict our attention to the $k = 0$ sector of the model. It may be mentioned here that the Calogero-Marchioro model in $D = 2$ is related to several systems of physical interest, e.g. quantum Hall effect, quantum dots, random matrix theory etc. [13]. Moreover, H_{CM} in $D = 2$ can be embedded into a Hamiltonian with extended $\mathcal{N} = 2$ superconformal symmetry [13].

The Hamiltonian in Eqn. (4.26) is translationally invariant and the parameter ν is determined from Eqn. (2.17) as

$$\nu = \frac{1}{2}(N-1)D - 1 + \frac{g}{2}N(N-1). \tag{4.28}$$

The many-body interactions are singular at the coincident positions of any two particles. If we do not want the wavefunction to have singularities of this kind, we have to impose the condition $g > 0$, which requires that $0 < \nu < 1$. Under this condition, for fixed values of N and D, H_{CM} admits self-adjoint extensions when

$$1 + \frac{2}{D} \leq N < 1 + \frac{4}{D}. \tag{4.29}$$

First note that the result of the A_{N-1} Calogero model is reproduced for $D = 1$. For $D = 2$, only $N = 2$ is allowed and for $D > 3$, there are no valid solution.

In the more general case, we allow the wavefunctions to have singularities while maintaining the square-integrability of the angular part coming from G. This puts the restriction that $g > -\frac{1}{2}$. In this case ν satisfies the constraint $-1 < \nu < 1$ which restricts g as

$$-\frac{D}{N} < g < -\frac{D}{N} + \frac{4}{N(N-1)}. \tag{4.30}$$

Let us first consider the case $\nu \neq 0$. The condition $g > -\frac{1}{2}$ is satisfied for

$$D < \frac{N}{2} + \frac{4}{N-1}. \tag{4.31}$$

The self-adjoint extension and consequently, the existence of scattering and bound states are admissible, provided Eqns. (4.30) and (4.31) are satisfied simultaneously. For $D > 3$, there exists a lower bound on N in order to have the self-adjoint extension. However, interestingly enough, for the physically important cases of $D = 1, 2$ and 3, the self-adjoint extensions are admissible for any arbitrary N.

When $\nu = 0$, the coupling g takes the value given by

$$g = \frac{2}{N(N-1)} - \frac{D}{N}. \tag{4.32}$$

The constraint $g > -\frac{1}{2}$ leads to the condition

$$D < \frac{2}{N-1} + \frac{N}{2}. \tag{4.33}$$

Note that Eqn. (4.33) is satisfied for arbitrary N when $D = 1, 2$. For $D = 3$, it is valid for $N \geq 6$. We end this discussion by noting that the wavefunctions and the spectrum for this model can be written explicitly as before, which we do not discuss here in detail.

5. Conclusion

In this paper we have discussed the quantization of N particle systems with classically scale invariant long range interactions. The effective Hamiltonian for these systems in the "radial" direction contains an inverse square term whose coefficient $(\nu^2 - \frac{1}{4})$ depends on the particle number N, the couplings and the "generalized angular momentum" k. It has been found that when $-1 < \nu < 1$, the Hamiltonians for this entire class of systems admit a one parameter family of self-adjoint extensions labeled by e^{iz} where $z \in R$ (mod 2π). The parameter z classifies all possible boundary conditions for which the Hamiltonian is self-adjoint. Moreover, the spectrum of the Hamiltonian depends explicitly on z. Each choice of the parameter z thus gives rise to an inequivalent quantization of the system.

We have illustrated the general approach through several examples which have been studied in detail. The examples in one dimension include A_{N-1} and B_N Calogero models without the confining term. The D_N Calogero model has been studied as a special case of the B_N case. In these cases we have found that a new class of bound and scattering states appear for arbitrary values of the particle number N and within certain ranges of the coupling constants. Moreover, these new states appear not only in the $k = 0$ sector, but for higher values of the quantum number k as well. To our knowledge, this is the first demonstration of the existence of self-adjoint extensions in the excited sectors of a system. This result may be attributed to the highly correlated nature of the many-body interaction. It may be mentioned that there is an important difference between the spectrum of $H_{A_{N-1}}$ and H_{B_N}. For fixed values of N and the relevant couplings, $H_{A_{N-1}}$ admits a maximum of two bound states corresponding to two allowed values of k. On the other hand, under similar conditions, B_N Calogero models admit only one bound state. This can be attributed to the fact that $H_{A_{N-1}}$ is translationally invariant, while H_{B_N} is not.

As an example of our general discussion in dimension higher than one, we have analyzed the Calogero-Marchioro model in D dimensions. In this case, the solution of the angular equations is known only for $k = 0$ and we have presented the analysis only for this sector. For the physically important cases of $D = 1, 2$ and 3, the new states are found to exist for arbitrary number of particles N and within certain range of the coupling constant.

Although the class of systems considered here is classically scale invariant, we have found that in presence of the self-adjoint extension, the systems may admit bound states. Moreover, the associated S matrix and the phase shifts are found to depend explicitly on the momentum. This happens as the classical scale invariance is broken due to quantization, which is manifest by the fact that the scaling operator does not leave the domain of self-adjointness of the Hamiltonian invariant. The analysis here thus provides further examples of quantum mechanical scaling anomaly. We have also shown that scale invariance at the quantum level can be implemented for special choices of the parameter z. It may also be mentioned in this context that the effective Hamiltonian $\tilde{H}$ in Eqn. (2.11) can be shown to be related to the Virasoro algebra [25] with central charge $c = 1$ [2, 7]. The systems studied here would also be related to the Virasoro algebra.

We have restricted our attention to the case where the coefficient of the inverse square potential is not too strongly negative in order to avoid the “fall to the center” [26]. The strong coupling region of the analogous 2-body problem has been analyzed in the literature using renormalization group techniques [27]. It would be interesting to consider the problem of renormalization in the presence of the self-adjoint extension. The N-particle rational Calogero models are also related to black holes [28] and Yang-Mills theories [10, 29]. It is plausible that the quantum states found here would have analogues in those cases as well. It would also be interesting to investigate the Calogero models associated with the exceptional Lie groups [9]. Finally we would like to mention that an analysis similar to the one presented in this paper can be done in the case of rational Calogero models in the presence of the confining term as well [30].

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